

# Complex Behavior in a Simple System of Classical Mechanics — Chaos

Newton's equation of motion is given by a second order ordinary differential equation

$$m \frac{d^2 x}{dt^2} = F(x(t), t). \quad (1)$$

In this chapter, first we learn how we can solve this type of differential equation numerically, and then actually solve one for a damped pendulum with external forcing. By doing it, we can see a phenomenon of chaos that shows a complex motion in apparently simple system.

## 1 Numerical Method to Solve Ordinary Differential Equation

Let us begin by solving the equation

$$\frac{dy}{dx} = f(x, y) \quad (2)$$

under the initial condition  $y(0) = y_0$ . Here,  $f(x, y)$  is a given function.

The problem is to find  $y$  as a function of  $x$ . To solve it numerically is to find the values for  $y(x)$  for a discrete set of  $x$

$$x = 0, h, 2h, 3h, \dots,$$

where  $h$  is a certain "small" number and we employ the notation

$$x_n \equiv nh, \quad y_n \equiv y(x_n)$$

in the following.

We are going to find  $y_1$  from  $y_0$ ,  $y_2$  from  $y_1$ , ...  $y_{i+1}$  from  $y_i$ , ... by iteration.

The simplest method is to use the equation

$$\begin{aligned} y_{i+1} &\equiv y(x_i + h) \\ &= y(x_i) + hy'(x_i) + O(h^2) \\ &= y_i + hf(x_i, y_i) + O(h^2), \end{aligned} \quad (3)$$

which is obtained from  $y_{i+1} = y(x_i + h)$  by Taylor expansion, and ignore the term of  $O(h^2)$ .

The method by eq.(3) is called **オイラー法 (Euler's method)**.

**問題 1.** Solve the ODE

$$\frac{dy}{dx} = \cos x; \quad y(0) = 0,$$

using Euler's method.

- (1) Take  $h$  as  $h = 1, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01$ , plot solutions for the range  $0 < x < 5$  for each  $h$ , and then compare them with the exact solution.
- (2) Examine how error from the exact solution at  $x = 5$  changes upon changing  $h$ . Try to plot errors against  $h$  in a way that you can see the  $h$ -dependence of the error. How you can understand the behavior from eq.(3)?

Euler's method is simple, but the precision improves only linearly with  $h$  when you use small  $h$ , thus it is not appropriate to use for a numerical solution in a good precision.

Now, we will try to improve Euler's method (3).

If eq.(3) for Euler's method is illustrated graphically, you can see the method approximates a curve by the tangential line using  $y_i$  and the slope  $y'_i \equiv f(x_i, y_i)$  at  $x_i$  in order to obtain

$y_{i+1}$  (Figure). Thus, in order to improve this, let us try to use the slope at the middle point  $x = x_{i+1/2} \equiv x_i + h/2$  instead of the slope at  $x_i$ , namely, employ the approximation

$$y_{i+1} \approx y_i + hy'(x_{i+1/2}),$$

instead of eq.(3). Here,

$$y'(x_{i+1/2}) = f(x_{i+1/2}, y_{i+1/2}),$$

is approximated as

$$y_{i+1/2} \approx y_i + \frac{1}{2}hf(x_i, y_i)$$

as in the case of eq.(3).

These are summarized as

$$y_{i+1} = y_i + k_1 \tag{4}$$

$$\begin{cases} k_0 = hf(x_i, y_i) \\ k_1 = hf(x_i + h/2, y_i + k_0/2) \end{cases} \tag{5}$$

The method by this set of equations is called **修正オイラー法 (Modified Euler's method)**.<sup>1</sup>

**問題 2.** Modified Euler's method:

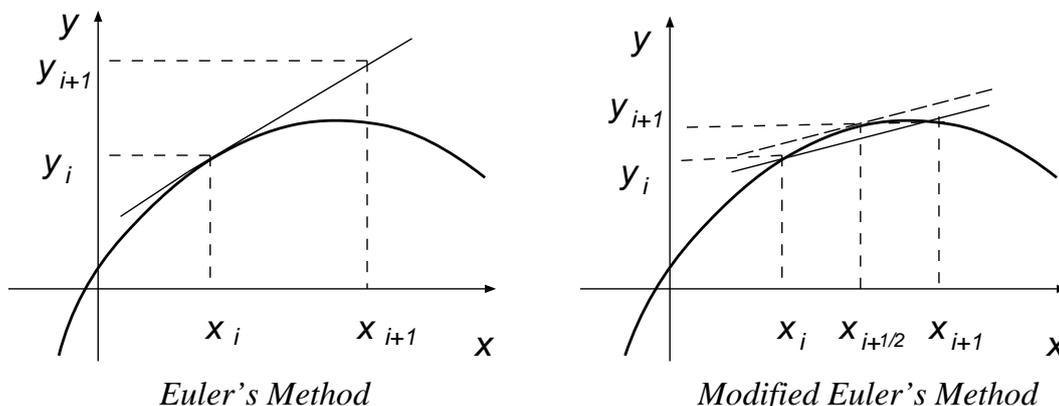
- (1) Repeat **問題 1**, using the modified Euler's method,
- (2) Estimate how much we can reduce the amount of computation needed to obtain a solution with the same precision by using the modified Euler's method in comparison with Euler's method.

In practice, the method called **ルンゲ・クッタ法 (Runge-Kutta method)** in the 4'th order is often used. It is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3) \tag{6}$$

$$\begin{cases} k_0 = hf(x_i, y_i) \\ k_1 = hf(x_i + h/2, y_i + k_0/2) \\ k_2 = hf(x_i + h/2, y_i + k_1/2) \\ k_3 = hf(x_i + h, y_i + k_2) \end{cases} \tag{7}$$

**問題 3.** Runge-Kutta method:



<sup>1</sup>Find more systematic derivation in an appropriate textbook.

- (1) Repeat 問題 2, using Runge-Kutta method.<sup>2</sup>
- (2) Find the derivation for eqs.(6) and (7) in an appropriate textbook.

So far, we have been dealing with ODE's of the 1'st order. How we can solve an ODE of the 2'nd order like an equation of motion

$$\frac{d^2y}{dx^2} = f(x, y, y'). \quad (8)$$

In such a case, we introduce a new variable  $u \equiv dy/dx$ , then eq.(8) can be expressed as a set of simultaneous equations with two variables

$$\begin{cases} \frac{dy}{dx} = u \\ \frac{du}{dx} = f(x, y, u) \end{cases}, \quad (9)$$

for which we can apply the previous methods.

**問題 4.** Write down the equations for Runge-Kutta method (6)–(7) in order to solve the ODE (8) of the 2'nd order.

[Hint] Rewrite Eq.(9) in a vector form as

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(x, \mathbf{y}); \quad \mathbf{y} \equiv \begin{pmatrix} y \\ u \end{pmatrix}, \quad \mathbf{f}(x, \mathbf{y}) \equiv \begin{pmatrix} u \\ f(x, y, u) \end{pmatrix},$$

and regard Eqs.(6)–(7) as vector equations. Then you can write down equations for their components.

## 2 Numerical Simulation for a Pendulum

Now, perform a numerical simulation for an actual physical system on a computer. First, consider an equation of motion for a harmonic oscillator

$$m \frac{d^2x}{dt^2} = -kx, \quad (10)$$

where  $m$  is a mass and  $k$  is a spring constant.

If you try to solve this as it is, you need to know actual values for  $m$  and  $k$ . However, if you scale the time  $t$  by the angular frequency  $\sqrt{k/m}$  and introduce a new variable  $\tau \equiv \sqrt{k/m} \cdot t$  to measure the time, then we have

$$\frac{d^2x}{d\tau^2} = -x, \quad (11)$$

thus, the quantities  $k$  and  $m$  do not appear in the equation. Namely, if we solve eq.(11), the solution of eq.(10) for different values of  $m$  and  $k$  can be obtained just by scaling the time.

It is very important to scale variables appropriately to remove apparent parameters before starting numerics.

**問題 5.** Solve eq.(11) using Runge-Kutta method under the initial condition  $\tau = 0 \text{ } \mathcal{C} \text{ } x = 0, dx/d\tau = 1$ .

- (1) Examine error from the exact solution by changing the step width  $h$  for  $\tau$ .
- (2) Examine how precisely the energy of the system is conserved numerically.<sup>3</sup>

Next, we consider a pendulum(Figure). The equation of motion is

$$ml\ddot{\theta} = -mg \sin \theta, \quad (12)$$

which can be written as

$$\frac{d^2\theta}{d\tau^2} = -\sin \theta \quad (13)$$

by scaling the time. It is well known that this can be approximated by a harmonic oscillator (11) for small oscillation, but in the case for large amplitude, eq.(13) is a non-linear equation, thus its solution cannot be expressed by elementary functions (in this case, the solution can be given by elliptic functions).

**問題 6.** Solve eq.(13) using Runge-Kutta method.

- (1) Confirm your program by checking the followings: (i) Confirm your numerical solution conserves energy. Check if the precision of the energy conservation improves as it should when you decrease  $h$ . (ii) Check if numerical solution for a given initial condition converges with an expected power of  $h$  as you decrease  $h$ .<sup>4</sup>
- (2) Plot  $\theta$  as a function of  $\tau$  for the initial conditions  $\theta = 0$  and  $d\theta/d\tau = 0.1, 0.5, 1, 1.9, 2, 2.1$  at  $\tau = 0$ .
- (3) Similarly, plot the trajectories of solution in the  $\theta$ - $\omega$  plane, where  $\omega \equiv d\theta/d\tau$ . Note that the range for  $\theta$  may be restricted within  $-\pi < \theta \leq \pi$  because  $\theta$  is an angle variable. (the  $\theta$ - $\omega$  plane is called **相空間 (Phase Space)**.)

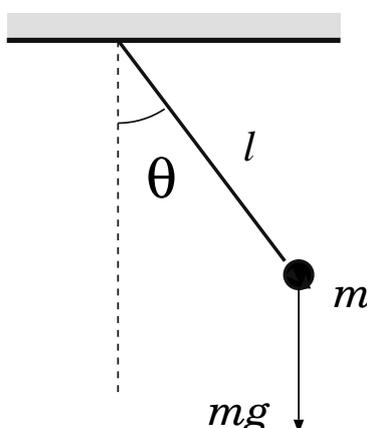
Now this pendulum is subjected to an resistive force proportional to the velocity and periodic external force, then the equatio of motion becomes

$$\frac{d^2\theta}{d\tau^2} = -\frac{1}{q} \frac{d\theta}{d\tau} - \sin \theta + g \cos \Omega\tau, \quad (14)$$

where  $q$  and  $g$  are parameters for the resistive and the external force, respectively, and  $\Omega$  is an angular frequency for the external force.

**問題 7.** Perform numerical simulation of the forced pendulum represented by the equation (14).

- (1) Plot trajectories of solutions in the phase space with arbitrary initial conditions. With  $q = 2$  and  $\Omega = 2/3$ , try the values of  $g$  as (a)  $g = 0.9$ , (b)  $g = 1.07$ , (c)  $g = 1.15$ , (d)  $g = 1.35$ . (e)  $g = 1.45$ , (f)  $g = 1.47$ , and (g)  $g = 1.50$ . After initial transient, a



<sup>2</sup>The error should decrease with  $h^4$  for small  $h$ .

<sup>3</sup>It is important to see how precisely a conserved quantity is actually conserved numerically, in order to check your program when an exact solution is not known.

<sup>4</sup>Use this method to check your program when you do not know any conserved quantity.

trajectory converges on the one that does not depend upon its initial condition. This trajectory is called **アトラクター (Attractor)**.

Plot the attractors by abandoning initial transient trajectories.<sup>5</sup>

- (2) Plot points in the phase space at every period of the external force, instead of plotting the trajectories continuously, i.e. plot  $(\theta, \omega)$  at  $\tau = 2\pi n/\Omega$  with  $n=0, 1, 2, 3, \dots$ . (Such a plot is called **ポアンカレ断面 (Poincaré Section)** of the attractor.)

For the cases of (c) and (g) in **問題 7**, you may find that the trajectories of the pendulum are very complicated. It is a bit surprising to find such complex behaviors in a simple system like forced pendulum.<sup>6</sup>

Actually, a phenomenon like this has been suggested quite a long time, but it is not until numerical simulations become easy by electronic computers that the phenomenon is recognized widely as **カオス (Chaos)**. Now, it has been known that such phenomena can be seen everywhere in nature.

As you can see from the Poincare section, the structure of attractors of chaotic trajectories is very complicated, thus called **ストレンジ・アトラクター (Strange Attractor)**. It can be said that *long term prediction is impossible* for phenomena that show chaotic trajectories.

In a sense, this is strange because a system such as the pendulum represented by eq.(14) is deterministic, thus the uniqueness of its solution is guaranteed if an initial condition is given, therefore, in principle, the motion of the pendulum is determined forever.

**問題 8.** Examine the behavior of trajectories on the attractors in the case of (c) and (g) of **問題 7**.

- (1) Pick a point  $P$  on the attractor, set it as an initial condition, and calculate the trajectory  $(\theta_P(\tau), \omega_P(\tau))$  for it.

Examine how the long term behavior of the trajectory converges as decreasing  $h$ .<sup>7</sup>

- (2) Calculate another trajectory  $(\theta_Q(\tau), \omega_Q(\tau))$  that starts from a point  $Q$  that is apart from the point  $P$  by a small distance, and compare it with the trajectory from  $P$ . Examine how the distance between the two trajectories

$$d(\tau) \equiv \sqrt{(\theta_P(\tau) - \theta_Q(\tau))^2 + (\omega_P(\tau) - \omega_Q(\tau))^2}$$

grows in time.

- (3) Examine the above (1) and (2) for the cases that do not show chaos.
- (4) From these observations, discuss the meaning of the statement “the long term prediction is impossible for a chaos trajectory”.

For a chaos trajectory, the distance between the two adjacent trajectories grows exponentially in time. Its exponent is called **リヤプノフ指数 (Lyapunov Exponent)**.

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<sup>5</sup>Length of initial transient depends upon values of parameters. Estimate the length of transient for each set of parameters.

<sup>6</sup>You never find such a phenomenon in a linear system.

<sup>7</sup>If you output all the data on a file for a long term trajectory, the size of the file will be huge and it could crash your computer. To avoid such situation, estimate the size of the data you are going to output before you perform simulation.